THE A_∞ OPERAD AND THE MODULI SPACE OF CURVES

KEVIN COSTELLO

Abstract. The modular envelope of a cyclic operad is the smallest modular operad containing it. A modular operad is constructed from moduli spaces of Riemann surfaces with boundary; this modular operad is shown to be the modular envelope of the A_∞ cyclic operad. This gives a new proof of the result of Harer-Mumford-Thurston-Penner-Kontsevich that a cell complex built from ribbon graphs is homotopy equivalent to the moduli space of curves.

1. Introduction

In recent years, an interesting relationship has emerged between the A_∞ (or associative) operad and the moduli spaces of curves. Probably the first person to make the connection to the associative operad was Witten [23, 24], who found the associativity relation appearing in open string theory. Around the same time, Harer-Mumford-Thurston [8] and Penner [18] showed that the cohomology of the moduli space of curves can be described by a complex built out of ribbon graphs. It was shown by Kontsevich [12] that ribbon graphs are closely related to the A_∞-operad; in particular he used ribbon graphs to associate to any A_∞-algebra certain cohomology classes in the moduli space of curves.

Several aspects of this story are a little unsatisfactory. For example, the results of Harer-Mumford-Thurston-Penner and Kontsevich rely on cell decompositions of the moduli space of curves, with a cell for each ribbon graph. However, there are many such triangulations known, with no canonical choice. A clear geometric reason for the description of the cohomology of moduli space by ribbon graphs seems to be lacking. Also, ribbon graphs form a (modular) operad – ribbon graphs can be glued along external edges. It was not clear (at least to me) what operadic structure on moduli space corresponds to this structure on ribbon graphs.

In another direction, it is known that a certain topological operad constructed from holomorphic discs with marked points on the boundary is isomorphic to the A_∞ topological operad. This is the reason for the appearance of A_∞ algebras in Floer homology and the Fukaya category [5, 4, 2, 3]. The associative operad appears in a closely related way in the work of Moore and Segal [17, 20]. They show that an open topological field theory, at all genera, is given by a (not necessarily commutative) Frobenius algebra. This is an analogue of the well-known result that a closed topological field theory is the same as a commutative Frobenius algebra.

This note describes a different point of view on the relationship between the A_∞ operad and the moduli space of curves, where all of the above results can be seen naturally. In particular, new proofs of the results of Kontsevich and Harer-Mumford-Thurston-Penner are given. The main result is that that the modular operad controlling open topological conformal field theory, at all genera, is the smallest modular operad containing the cyclic operad of A_∞ algebras. One immediate corollary is that an open topological conformal field theory is the same as an A_∞ algebra with invariant inner product. This is a generalisation of the work of Moore and Segal on open topological field theory. The distinction between topological field theory and topological conformal field theory is that the former deals with topological surfaces, while the latter takes

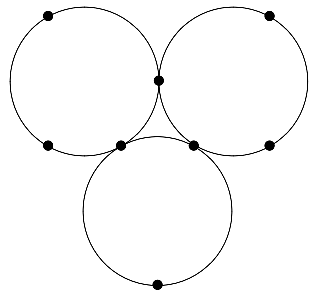


FIGURE 1. A point in $\overline{\mathcal{N}}_{0,2,5}$ corresponding to three discs glued together

account of the topology of moduli spaces of conformal structures on surfaces. Another corollary of the main result is the ribbon graph decomposition of moduli space.

Moduli spaces $\overline{\mathcal{N}}_{g,n,r}$ of Riemann surfaces with boundary, and with marked points and possibly nodes on the boundary, are described. These moduli spaces are manifolds with corners; the boundary $\partial\overline{\mathcal{N}}_{g,n,r}$ is the locus of singular surfaces. These moduli spaces are close relatives of those studied by Liu and Katz [13, 9] and Fukaya-Oh-Ohta-Ono [5]; they are an open subset of the natural Deligne-Mumford compactification of the moduli of Riemann surfaces with boundary, and are closely related to the real points of the usual Deligne-Mumford spaces.

The topological type of a curve C in the Deligne-Mumford moduli space can be described by a graph. In a similar way, the topological type of a Riemann surface with nodal boundary $\Sigma \in \overline{\mathcal{N}}_{g,n,r}$ can be described by a type of graph. If all of the irreducible components of $\Sigma \in \overline{\mathcal{N}}_{g,n,r}$ are discs, then the topological type of Σ is described by a ribbon graph, with r external edges. Let $D_{g,n,r} \hookrightarrow \overline{\mathcal{N}}_{g,n,r}$ be the locus of such surfaces. I show that the inclusion $D_{g,n,r} \hookrightarrow \overline{\mathcal{N}}_{g,n,r}$ is a homotopy equivalence. $\overline{\mathcal{N}}_{g,n,r}$ is a manifold with corners, and so is homotopy equivalent to its interior $\mathcal{N}_{g,n,r}$. When $r = 0$, $\mathcal{N}_{g,n,0}$ is homotopy equivalent to $\mathcal{M}_{g,n}$, the space of smooth complex algebraic curves with n marked points. Therefore there is a homotopy equivalence $D_{g,n,0} \simeq \mathcal{M}_{g,n}$. The locus in $D_{g,n,0}$ of surfaces whose topological type is described by a fixed ribbon graph is an orbi-cell. This immediately implies that the complex of singular chains $C_*(\mathcal{M}_{g,n} \otimes \mathbb{Q})$ is quasi-isomorphic to a complex built from ribbon graphs, recovering the results of Harer-Mumford-Thurston-Penner.

The spaces $\overline{\mathcal{N}}_{g,n,r}$ form a modular operad, by gluing surfaces along marked points. The statement that $\overline{\mathcal{N}}_{g,n,r}$ is homotopy equivalent to $D_{g,n,r}$ can be interpreted as saying that this operad is generated (up to homotopy) by the moduli spaces $\overline{\mathcal{N}}_{0,1,r}$ of discs. All the relations also come from $\overline{\mathcal{N}}_{0,1,r}$. The moduli space $\overline{\mathcal{N}}_{0,1,r}$ of discs form a cyclic operad - if two discs are glued together, the result is still a (singular) disc. This cyclic operad is isomorphic to the topological cyclic operad A_∞^{top} of Stasheff [21].

To make this statement precise I need to develop some operadic formalism. The notions of cyclic and modular operads, which are generalizations of operad, were introduced by Getzler and Kapranov [6, 7], following Kontsevich's work on graph cohomology [11, 12]. A good introduction to these operadic concepts can be found in the book of Markl, Shnider and Stasheff [16]. Roughly, a cyclic (resp. modular) operad is to a forest (resp. graph) what an operad is to a rooted forest (recall a forest is a disjoint union of trees). This can be made precise: I define symmetric monoidal categories **RForests**, **Forests** and **Graphs**, whose morphisms are given by rooted forests, forests, and graphs. If **C** is a symmetric monoidal category, a tensor functor **RForests** $\to$ **C** (resp. **Forests** $\to$ **C**, **Graphs** $\to$ **C**) is the same as an operad (resp. cyclic

operad, modular operad) in $\mathbf{C}$. There is a functor $\mathbf{Forests} \to \mathbf{Graphs}$; therefore every modular operad is also a cyclic operad. Conversely, for every cyclic operad P, I define a modular operad $\mathbf{Mod}(P)$, the "modular envelope" of P (by analogy with the universal enveloping algebra of a Lie algebra). $\mathbf{Mod}(P)$ comes equipped with a map of cyclic operads $P \to \mathbf{Mod}(P)$. This map is universal, in the sense that if Q is a modular operad and $P \to Q$ is a map of cyclic operads, there is a unique map of modular operads $\mathbf{Mod}(P) \to Q$ making the obvious diagram commute.

The moduli spaces $\overline{\mathcal{N}}_{g,n,r}$ form a topological modular operad $\overline{\mathcal{N}}$, by gluing surfaces at marked points. The moduli spaces $\overline{\mathcal{N}}_{0,1,r}$ form a cyclic operad, by gluing two discs along marked points to get a singular disc. It is known that this cyclic operad is isomorphic to the topological operad A_∞^{top} of Stasheff [21]. Thus there is a map of topological cyclic operads, $A_\infty^{top} \to \overline{\mathcal{N}}$. By the universal property of the modular envelope construction, this induces a map $\mathbf{Mod}(A_\infty^{top}) \to \overline{\mathcal{N}}$. The main result of this paper then says

Theorem 1.0.1. *The map $\mathbf{Mod}(A_\infty^{top}) \to \overline{\mathcal{N}}$ is a homotopy equivalence of orbispace modular operads.*

Let C_* be an appropriate chain-complex functor from topological spaces to dg $\mathbb{Q}$–vector spaces. Applying C_* turns a topological operad into a dg operad; this gives a dg modular operad $C_*(\overline{\mathcal{N}})$ and a dg cyclic operad $C_*(A_\infty^{top})$. It is known that $C_*(A_\infty^{top})$ is quasi-isomorphic to the usual algebraic A_∞ operad, A_∞^{alg}.

Corollary 1.0.2. *There is a quasi-isomorphism of dg modular operads*

$$\mathbf{Mod}(A_\infty^{alg} \otimes \mathbb{Q}) \to C_*(\overline{\mathcal{N}})$$

The operad $\overline{\mathcal{N}}$ is the operad controlling open topological conformal field theory (TCFT); an algebra over the modular operad $C_*(\overline{\mathcal{N}})$ is called an open TCFT. [1] The reason for this terminology is as follows. Recall that the operad structure on $\overline{\mathcal{N}}$ is given by gluing marked points $P_1 \in \partial\Sigma_1$, $P_2 \in \partial\Sigma_2$ together. Replace each point P_i by a small interval I_i; this gives us a homotopy equivalent moduli space. The operad structure is given by gluing these intervals together. This is like the string theory operation of gluing an incoming open string to an outgoing open string.

Applying π_0 to the homotopy equivalence $\mathbf{Mod}(A_\infty^{top}) \simeq \overline{\mathcal{N}}$ recovers the result of Moore and Segal relating topological open field theory to Frobenius algebras. This is because $\pi_0(\mathbf{Mod}(A_\infty^{top}))$ is the modular envelope of the associative cyclic operad; whereas $\pi_0(\overline{\mathcal{N}})$ is a modular operad in the category of sets, constructed from isomorphism classes of topological surfaces with marked intervals on the boundary. The result of Moore and Segal can be interpreted as saying that algebras over the modular operad $\pi_0(\overline{\mathcal{N}})$ are precisely associative algebras with inner product, or in other terms that the operad $\pi_0(\overline{\mathcal{N}})$ is isomorphic to $\mathbf{Mod}(\text{Assoc})$.

The main result of this note is therefore a close analogue of this theorem of Moore and Segal. In fact, what is proved here is a "derived" version of their result. As Ezra Getzler pointed out to me, one can interpret the modular operad $\mathbf{Mod}(A_\infty^{alg})$ as being $\mathbb{L}\mathbf{Mod}(\text{Assoc})$, where $\mathbb{L}\mathbf{Mod}$ is the left derived functor in the sense of homotopical algebra. This is because the operad A_∞^{alg} is a free resolution (and therefore a cofibrant model) of the operad Assoc of associative algebras. Therefore, what is shown here is that $\mathbb{L}\mathbf{Mod}(\text{Assoc} \otimes \mathbb{Q}) \cong C_*(\overline{\mathcal{N}})$. One would hope that in a similar way, the topological A_∞ operad is a cofibrant model (whatever that means) of the associative cyclic operad, in the category of topological cyclic operads; this would show that

[1] This is not the most general definition of open TCFT; if we generalise to allow more than one D-brane, we find an open TCFT is the same as an A_∞ category of Calabi-Yau type.

$\mathbb{L}\mathbf{Mod}(\mathrm{Assoc}) \cong \overline{\mathcal{N}}$. Of course there are considerable difficulties making sense of this in the topological setting.

$A_\infty^{alg} \otimes \mathbb{Q}$ is not just a cofibrant model of the associative operad over $\mathbb{Q}$; it is the minimal model in the sense of Markl [15]. Markl's theory should generalise without much difficulty to modular and cyclic operads. Then $\mathbf{Mod}(A_\infty^{alg} \otimes \mathbb{Q})$ is the minimal model of $C_*(\overline{\mathcal{N}}) \otimes \mathbb{Q}$. As, $\mathbf{Mod}(A_\infty^{alg} \otimes \mathbb{Q})$ is free (after forgetting the differential), and the image of the differential is in the space of decomposable elements. If we define a homotopy action of a connected modular operad on a complex to be an action of its minimal model, this should show that a homotopy action of $C_*(\overline{\mathcal{N}}) \otimes \mathbb{Q}$ on a complex V with inner product is the same as an action of $\mathbf{Mod}(A_\infty^{alg} \otimes \mathbb{Q})$ on V. However, this is the same as an A_∞ structure on V - that is a homotopy associative structure. Therefore, we see that an open TCFT (at all genera) is precisely a homotopy associative Frobenius algebra.

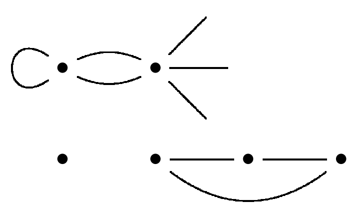
I should remark that the operads $\overline{\mathcal{N}}_{g,n,r}$ seem to be related to the arc operads studied by Penner and Kauffmann, Livernet, Penner in the interesting papers [19, 10]. However, they are mostly concerned with the operads of closed rather than open strings.

Acknowledgements. I would like to thank Ezra Getzler and Jim Stasheff for their comments on an earlier version of this note, and Graeme Segal for patiently explaining his work to me.

2. Cyclic and modular operads

In this section, I give a definition of operads, cyclic operads, and modular operads. The definitions presented here have a slightly different flavour to the usual definitions (although they are essentially equivalent). In this note, an operad (resp. cyclic operad, modular operad) in a symmetric monoidal category $\mathbf{C}$ is defined to be a tensor functor from a symmetric monoidal category constructed from rooted forests (resp. forests, graphs) to $\mathbf{C}$. The advantage of this definition is that it makes the construction of free operads, and of the modular envelope of a cyclic operad, easier. I begin by defining the categories **RForests**, **Forests**, **Graphs**.

A graph is what you think it is : it is a collection of vertices joined by edges. Graphs may be disconnected, there may be external edges (or tails), and vertices may have loops. Slightly more degenerate graphs are allowed than is usual: for example, there is a graph with one vertex and no edges. Here is a picture of a graph.



There are various finite sets associated to a graph γ. There is the set $C(\gamma)$ of connected components, the set $T(\gamma)$ of tails or external edges, the set $V(\gamma)$ of vertices, and the set $H(\gamma)$ of germs of edges, or half-edges. A half-edge is an edge (internal or external) together with the choice of a vertex attached to it. There are maps $T(\gamma) \to C(\gamma)$ and $H(\gamma) \to V(\gamma)$. For a vertex v, write $H(v)$ for the fibre of $H(\gamma) \to V(\gamma)$ at v.

For example, in the picture above, $\#C(\gamma) = 3$, $\#T(\gamma) = 3$, $\#V(\gamma) = 6$ and $\#H(\gamma) = 15$. If v is the vertex on the upper left, then $\#H(v) = 4$.

It is convenient to give a formal definition of a graph.

Definition 2.0.3. *A* graph γ *consists of*

- *A finite set* $V(\gamma)$ *of vertices;*

- *a finite set $H(\gamma)$ of half-edges (or germs of edges);*
- *a map $\pi : H(\gamma) \to V(\gamma)$;*
- *an involution $\sigma : H(\gamma) \to H(\gamma)$, satisfying $\sigma^2 = 1$.*

The edges *$E(\gamma)$ of γ are the free σ-orbits in $H(\gamma)$. The* tails *$T(\gamma)$ of γ are the σ fixed points of $H(\gamma)$. The set of connected components $C(\gamma)$ of γ is the quotient of $V(\gamma)$ by the equivalence relation generated by : $v \sim v'$ if there is a half-edge $h \in H(\gamma)$, with $\pi(h) = v$ and $\pi(\sigma(h)) = v'$. There is a map $T(\gamma) \to C(\gamma)$.*

The geometric realization *$|\gamma|$ of γ is the cell complex, with a 0-cell for each vertex $v \in V(\gamma)$, and a copy I_h the interval $I = [0, 1]$ for each half-edge $h \in H(\gamma)$. $0 \in I_h$ is glued to the vertex $\pi(h)$, I_h is identified with $I_{\sigma(h)}$ via the map $I_h \to I_{\sigma(h)}$, $t \to 1 - t$.*

A forest *is a graph all of whose connected components are contractible, and each of whose vertices is at least trivalent. A* rooted forest *is a forest together with a choice of tail for each connected component, that is a section $C(\gamma) \to T(\gamma)$ of the projection $T(\gamma) \to C(\gamma)$.*

Now I can define the categories **Graphs**, **Forests** and **RForests**. An object of **Graphs** is a pair I, J of finite sets, together with a map $I \to J$. In order to distinguish these maps of finite sets from the morphisms in the category **Graphs**, I will write $[I \twoheadrightarrow J]$ for this object of **Graphs**.

The morphisms of **Graphs** are given by graphs. A graph γ is a morphism

$$\gamma : [H(\gamma) \twoheadrightarrow V(\gamma)] \to [T(\gamma) \twoheadrightarrow C(\gamma)].$$

Let γ_1, γ_2 be graphs, with an isomorphism

$$[T(\gamma_2) \twoheadrightarrow C(\gamma_2)] \cong [H(\gamma_1) \twoheadrightarrow V(\gamma_1)].$$

Thus, γ_2 is a morphism

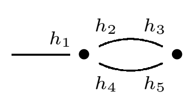
$$\gamma_2 : [H(\gamma_2) \twoheadrightarrow V(\gamma_2)] \to [T(\gamma_2) \twoheadrightarrow C(\gamma_2)],$$

and γ_1 is a morphism

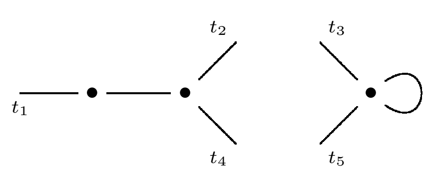
$$\gamma_1 : [T(\gamma_2) \twoheadrightarrow V(\gamma_2)] \to [T(\gamma_1) \twoheadrightarrow C(\gamma_1)].$$

The composition $\gamma_1 \circ \gamma_2$ can be formed by inserting γ_2 into γ_1. That is, each vertex $v \in V(\gamma_1)$ is replaced by the corresponding connected component of γ_2 under the identification $V(\gamma_1) \cong C(\gamma_2)$, and the half-edges $H(v)$ are glued at v to the corresponding tails of the connected component of γ_2, using the identification $H(\gamma_1) \cong T(\gamma_2)$.

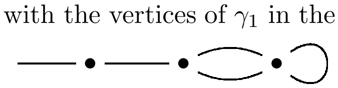
The composition of two graphs can be drawn as follows. Let γ_1 be the graph



and let γ_2 be the graph



In order to form the composition $\gamma_1 \circ \gamma_2$, we need to identify the tails $T(\gamma_2)$ with the half-edges $H(\gamma_1)$. The tail $t_i \in T(\gamma_2)$ is identified with the half-edge $h_i \in H(\gamma_1)$. The connected components of γ_2 are identified with the vertices of γ_1 in the only possible way. Then $\gamma_1 \circ \gamma_2$ is



If $[I \twoheadrightarrow J]$ is an object of **Graphs**, the identity map $[I \twoheadrightarrow J]$ is given by the graph γ with $T(\gamma) = H(\gamma) = I$ and $V(\gamma) = C(\gamma) = J$. The involution $H(\gamma) \to H(\gamma)$ is the identity, or in other terms γ has no internal edges. If $[I \twoheadrightarrow J]$, $[K \twoheadrightarrow L]$ are two objects of **Graphs**, an isomorphism (of pairs of finite sets with morphisms) $[I \twoheadrightarrow J] \cong [K \twoheadrightarrow L]$ corresponds to the graph γ with $H(\gamma) = I$, $V(\gamma) = J$, and no internal edges as before; but with the isomorphism $T(\gamma) \cong K$, $C(\gamma) \cong L$.

Graphs is a symmetric monoidal category. The tensor product is defined on objects by

$$[I \twoheadrightarrow J] \otimes [K \twoheadrightarrow L] = \left[(I \coprod K) \twoheadrightarrow (J \coprod L)\right]$$

and on morphisms by

$$\gamma_1 \otimes \gamma_2 = \gamma_1 \coprod \gamma_2.$$

Let **Forests** $\subset$ **Graphs** be the subcategory such that Ob **Forests** $\subset \operatorname{Ob}$ **Graphs** consists of those $[I \twoheadrightarrow J] \in \operatorname{Ob}$ **Graphs** such that the fibres of the map of finite sets $I \twoheadrightarrow J$ are of cardinality at least 3. Mor **Forests** $\subset \operatorname{Mor}$ **Graphs** consists of forests. **Forests** is a symmetric monoidal category.

Let **RForests** be the category whose objects are maps $I \twoheadrightarrow J$ of finite sets such that the cardinality of the fibres is at least 3, together with a section $\sigma : J \to I$. The morphisms in **RForests** are given by rooted forests. The composition **RForests** is defined in a similar way to that in **Graphs**. There is a functor **RForests** $\to$ **Forests**.

After these preliminaries, I can define the types of operad I need.

Definition 2.0.4. *Let* **C** *be a symmetric monoidal category. A* modular operad *is a tensor functor*

$$P : \mathbf{Graphs} \to \mathbf{C}.$$

A cyclic operad *is a tensor functor* **Forests** $\to$ **C**. *An* operad *is a tensor functor* **RForests** $\to$ **C**.

One can see that these definitions are equivalent to the more traditional definitions. For example, any tensor functor F : **RForests** $\to$ **C** is determined by its action on the objects $[(I \cup \{*\}) \twoheadrightarrow \{*\}]$, and on the morphisms corresponding to connected rooted forests with one internal edge. This is because these objects and morphisms generate **RForests** as a symmetric monoidal category. For each finite set I, let

$$P(I) = F([(I \cup \{*\}) \twoheadrightarrow \{*\}]) \in \operatorname{Ob}\mathbf{C}$$

$P(I)$ is acted on by $\operatorname{Aut} I$. For each pair I, J of finite sets, with elements $i \in I$, there is a rooted forest in Mor **RForests** with one internal edge, which gives a morphism

$$[(I \cup \{*\}) \twoheadrightarrow \{*\}] \otimes [(J \cup \{*\}) \twoheadrightarrow \{*\}] \to [(I \setminus \{i\} \cup J \cup \{*\}) \twoheadrightarrow \{*\}]$$

This shows that a tensor functor **RForests** $\to$ **C** is the same as a collection of objects $P(I) \in \mathbf{C}$, one for each finite set I with at least 3 elements, together with an $\operatorname{Aut}(I)$ action on $P(I)$; and for each finite set J, and element $i \in I$, a composition map

$$P(I) \otimes P(J) \to P(I \setminus \{i\} \cup J)$$

This composition map must satisfy a certain associativity property; and we find that a tensor functor **RForests** $\to$ **C** is the same as an operad in **C**.

Similar remarks hold for cyclic and modular operads. For any functor P : **Graphs** $\to$ **C** or **Forests** $\to$ **C**, and any finite set I, we will abuse notation and write

$$P(I) = P([I \twoheadrightarrow *])$$

If $n \in \mathbb{Z}_{\geq 0}$, we will also write $P(n)$ for $P([n])$ where $[n] = \{1, 2, \ldots, n\}$. For $i \in I, j \in J$, we will write

$$\circ_{i,j} : P(I) \otimes P(J) \to P(I \cup J \setminus \{i, j\})$$

for the composition map coming from the morphism in **Forests** or **Graphs**,

$$[I \cup J \twoheadrightarrow \{1, 2\}] \to [I \cup J \setminus \{i, j\} \twoheadrightarrow \{*\}]$$

If P is a functor **Graphs** $\to$ **C**, and $i_1, i_2 \in I$ are distinct, we will write

$$\circ_{i_1,i_2} : P(I) \to P(I \setminus \{i_1, i_2\})$$

for the map coming from the morphism in **Graphs**,

$$[I \twoheadrightarrow \{*\}] \to [I \setminus \{i_1, i_2\} \twoheadrightarrow \{*\}]$$

It is easy to see that the composition maps $\circ_{i,j}$ and $\circ_{i_1,i_2}$ satisfy the usual associativity and equivariance axioms for a cyclic or modular operad. Also P : **Graphs** $\to$ **C** or **Forests** $\to$ **C** is completely determined by the $P(I)$, with their natural $\mathrm{Aut}(I)$ action, and these composition maps.

2.1. **Free operads and the modular envelope.** Suppose $\mathbf{C}_1$, $\mathbf{C}_2$, $\mathbf{C}_3$, are categories, and $F : \mathbf{C}_1 \to \mathbf{C}_2$ is a functor. There is a pull-back functor

$$F^* : \mathrm{Fun}(\mathbf{C_2}, \mathbf{C_3}) \to \mathrm{Fun}(\mathbf{C_1}, \mathbf{C_3})$$

given by composition with F. In certain nice situations, this functor admits a left adjoint

$$F_* : \mathrm{Fun}(\mathbf{C_1}, \mathbf{C_3}) \to \mathrm{Fun}(\mathbf{C_2}, \mathbf{C_3})$$

The construction of F_* is an analogue of the familiar induction of group representations. If $G : \mathbf{C_1} \to \mathbf{C_3}$ is a functor, then F_*G satisfies a universal property. There is a morphism of functors

$$G \to F^* F_*(G) = F_*(G) \circ F$$

such that if $H : \mathbf{C}_2 \to \mathbf{C}_3$ is any functor, and $G \to F^* H$ is a morphism of functors, then there is a unique morphism of functors $F_* G \to H$ such that the diagram

$$\begin{tikzcd} G \arrow[r] \arrow[dr] & F^* F_* G \arrow[d] \\ & F^* H \end{tikzcd}$$

commutes.

A functor F_*G with this universal property doesn't always exist; it only exists when $\mathbf{C}_3$ admits enough coproducts. In this case, it is defined as follows. For an object $x \in \mathbf{C}_2$, $F_*G(x)$ has a copy $G(y)_f$ of $G(y)$ for each object $y \in \mathrm{Ob}\,\mathbf{C}_1$ with a morphism $f : F(y) \to x$. If $g : y' \to y$ is a morphism in $\mathbf{C_1}$, then the copy $G(y')_{f \circ F(g)}$ of $G(y')$ is identified with the image of $G(g) : G(y') \to G(y) = G(y)_f$. F_*, when it exists, is the left adjoint to the functor $F^* : \mathrm{Fun}(C_2, C_3) \to \mathrm{Fun}(C_1, C_3)$ given by composition with F.

This construction will be applied to construct free modular and cyclic operads, and to construct a modular operad from a cyclic operad.

Let $\mathbf{C}$ be a symmetric monoidal category, which is now assumed to be k-linear. Let **Pairs** be the tensor category whose objects are the same as those of **Graphs**, that is pairs I, J of finite sets with a map $I \twoheadrightarrow J$. The morphisms in **Pairs** are simply isomorphisms $[I \twoheadrightarrow J] \cong [I' \twoheadrightarrow J']$, so that **Pairs** is a groupoid. The tensor structure on **Pairs** is given by disjoint union, as before. Let P : **Pairs** $\to$ **C** be a tensor functor. There is a functor F : **Pairs** $\to$ **Graphs**; in fact **Pairs** is the subcategory of **Graphs** whose objects are all those of **Graphs** and whose morphisms are

the isomorphisms in **Graphs**. $F_* P$ is a functor $\mathbf{Graphs} \to \mathbf{C}$, that is a modular operad. $F_* P$ is the free modular operad generated by P.

Now suppose $P : \mathbf{Forests} \to \mathbf{C}$ is a cyclic operad. Since their is a functor $\Gamma : \mathbf{Forests} \to \mathbf{Graphs}$, every modular operad is in particular a cyclic operad. That is, there is a forgetful functor Γ^* from modular to cyclic operads. Denote this functor by **Cyc**. There is a left adjoint Γ_* to this functor, as long as $\mathbf{C}$ admits enough coproducts. Denote this functor by **Mod**.

I call $\mathbf{Mod}(P)$ the *modular envelope* of P, by analogy with the universal enveloping algebra of a Lie algebra. The modular envelope satisfies a universal property. There is a map $P \to \mathbf{Cyc}(\mathbf{Mod}(P))$ of cyclic operads. For any modular operad Q with a map $P \to \mathbf{Cyc}(Q)$, there is a unique map $\mathbf{Mod}(P) \to Q$ such that the diagram

$$\begin{tikzcd} P \arrow[r] \arrow[dr] & \mathbf{Cyc}(\mathbf{Mod}(P)) \arrow[d] \\ & \mathbf{Cyc}(Q) \end{tikzcd}$$

commutes.

Note that there is no problem in extending the definition of cyclic and modular operads to the 2-category of topological orbispaces. Let **Orb** be this 2 category; there is an obvious functor $F : \mathbf{Top} \to \mathbf{Orb}$. Then, if $P : \mathbf{Forests} \to \mathbf{Top}$ is a cyclic operad in the category of topological spaces, $F \circ P : \mathbf{Forests} \to \mathbf{Orb}$ is a cyclic operad in **Orb**. Form the modular envelops $\mathbf{Mod}(P) : \mathbf{Graphs} \to \mathbf{Top}$, the modular envelope of P, and $\mathbf{Mod}(F \circ P)$, the modular envelope of $F \circ P$. These are not the same, that is $F \circ \mathbf{Mod}(P) \neq \mathbf{Mod}(F \circ P)$. In many ways it is better to consider $\mathbf{Mod}(F \circ P)$, because the construction of **Mod** involves forming quotients by actions of finite groups.

2.2. **Examples of cyclic operads.** The first example is $\mathcal{A}$, the associative cyclic operad; this is a cyclic operad in the tensor category **Fin** of finite sets, whose morphisms are isomorphisms of finite sets. The tensor structure on **Fin** is given by Cartesian product.

For an object $[I \twoheadrightarrow J] \in \operatorname{Ob} \mathbf{Forests}$, define

$$\mathcal{A}([I \twoheadrightarrow J]) = \{\text{cyclic orders on the fibres of } I \twoheadrightarrow J\}$$

For a morphism $\gamma : [H(\gamma) \twoheadrightarrow V(\gamma)] \to [T(\gamma) \twoheadrightarrow C(\gamma)]$, we need to define

$$\mathcal{A}(\gamma) : \mathcal{A}([H(\gamma) \twoheadrightarrow V(\gamma)]) \to \mathcal{A}([T(\gamma) \twoheadrightarrow V(\gamma)])$$

Note that an element $a \in \mathcal{A}([H(\gamma) \twoheadrightarrow V(\gamma)])$ corresponds to a cyclic order on the set of edges emanating from each vertex of the forest γ. That is, γ becomes a ribbon graph. γ can be thickened to form a compact oriented surface Σ with boundary; as γ is a forest, Σ is a disjoint union of discs. The orientation on $\partial\Sigma$ induced by that on Σ induces a cyclic order on the tails $T(c)$ for each connected component $c \in C(\gamma)$, that is a cyclic order on the fibres of $T(\gamma) \to C(\gamma)$; this defines $\mathcal{A}(\gamma)$.

$\mathcal{A}$ is called the associative cyclic operad. There is a tensor functor $\mathbf{Fin} \to \mathbf{Vect}_k$ which sends a finite set S to the vector space $k^{\oplus S}$ with basis S. The cyclic operad $\mathcal{A}$ in **Fin** pushes forward to the usual cyclic operad of associative algebras in the category of vector spaces, which I also denote by $\mathcal{A}$.

The second basic example is $\mathcal{C}$, the cyclic operad of commutative algebras. Again, $\mathcal{C}$ is a cyclic operad in the tensor category **Fin**. For an object $a \in \operatorname{Ob} \mathbf{Forests}$, define $\mathcal{C}(a) = \{*\}$, the set with one element. The definition of $\mathcal{C}$ on morphisms in **Forests** is trivial.

In this paper, the cyclic operad we will be most concerned with is A_∞^{alg}, the operad of A_∞ algebras. A_∞^{alg} is a cyclic operad in the category of differential graded $\mathbb{Q}$ vector spaces, with

differential of degree -1. (This choice of degree of the differential is so as to be consistent with the choice for chain complexes of topological spaces later on). Let $\mathbf{dg}_{\mathbb{Q}}$ be this tensor category, and let $\mathbf{Vect}_{\mathbb{Q}}^{\mathbb{Z}}$ be the category of graded $\mathbb{Q}$ vector spaces. As a graded cyclic operad, that is forgetting the differential, A_∞^{alg} is freely generated by a certain tensor functor $F : \mathrm{Fin} \to \mathbf{Vect}_{\mathbb{Q}}^{\mathbb{Z}}$. On the finite set $[I \twoheadrightarrow *]$, F is defined to be the $\mathbb{Q}$ vector space with basis the set of cyclic orders on I, situated in degree $3 - \#I$. The functor F is extended to **Fin** by making it a tensor functor.

Taking the free cyclic operad on F gives us a tensor functor $A_{\infty\#}^{alg} : \mathbf{Forests} \to \mathbf{Vect}_{\mathbb{Q}}^{\mathbb{Z}}$. The cyclic operad A_∞^{alg} is obtained from this by adjoining a certain differential. If $[I \twoheadrightarrow J] \in \mathrm{Ob}\,\mathbf{Forests}$, then $A_{\infty\#}^{alg}([I \twoheadrightarrow J])$ has a basis corresponding to forests γ with isomorphisms $[T(\gamma) \twoheadrightarrow C(\gamma)] \cong [I \twoheadrightarrow J]$, with cyclic orders on the fibres of $[H(\gamma) \twoheadrightarrow V(\gamma)]$, and with orderings of the set $V(\gamma)$. Reordering the set $V(\gamma)$ changes the basis element by a sign corresponding to the order of the permutation. The basis element corresponding to a forest γ has degree $\#H(\gamma) - 3\#V(\gamma)$. The differential is defined on these basis elements by summing over all ways to add an edge to the forest, with appropriate sign.

2.3. **Algebras over cyclic and modular operads.** Let P be a cyclic or modular operad in the tensor category $\mathbf{C}$; $\mathbf{C}$ is now assumed to be one of the categories of finite dimensional vector spaces, $\mathbb{Z}$-graded vector spaces or dg vector spaces.

I want to define the notion of a P-action on an object $V \in \mathrm{Ob}\,\mathbf{C}$. Suppose $\triangle \in V^{\otimes 2}$ is a closed element of degree 0 say. Then I define a modular operad $\mathrm{End}(V, \triangle)$, as follows. For an object $[I \twoheadrightarrow J] \in \mathrm{Ob}\,\mathbf{Graphs}$, define

$$\mathrm{End}(V, \triangle)([I \twoheadrightarrow J]) = V^{\vee \otimes I}$$

The modular operad structure on $\mathrm{End}(V, \triangle)$ uses the tensor $\triangle$. If $\gamma : [H(\gamma) \twoheadrightarrow V(\gamma)] \to [T(\gamma) \twoheadrightarrow C(\gamma)]$ is a morphism in **Graphs**, then define

$$\mathrm{End}(V, \triangle)(\gamma) : V^{\vee \otimes H(\gamma)} \to V^{\vee \otimes T(\gamma)}$$

to be

$$\mathrm{End}(V, \triangle)(\gamma) = \otimes_{e \in E(\gamma)} \triangle_e$$

that is the tensor product of a copy of $\triangle$ for each edge $e \in E(\gamma)$, acting on the half-edges corresponding to e. With these definitions, it is easy to see that the composition maps are of degree 0.

If P is a cyclic operad, then a P action on $(V, \triangle)$ is a map of cyclic operads

$$P \to \mathbf{Cyc}\,(\mathrm{End}(V, \triangle))$$

If P is a modular operad, a P action on $(V, \triangle)$ is a map of modular operads

$$P \to \mathrm{End}(V, \triangle)$$

Note that if P is a cyclic operad, then a P action on $(V, \triangle)$ is the same as a $\mathbf{Mod}(P)$ action on $(V, \triangle)$.

The definition of action of a cyclic operad on a complex presented here is possibly too restrictive; for interesting work on generalising the notion of an algebra over a cyclic operad see [22, 14].

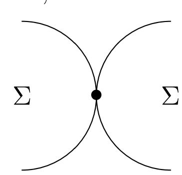
3. The open TCFT operad and the A_∞ operad

3.1. **Recollections on Riemann surfaces with boundary.** A Riemann surface of genus g with $n > 0$ boundary components has the following equivalent descriptions.

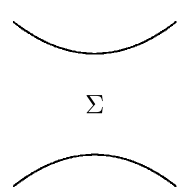
(1) A compact connected ringed space Σ, isomorphic as a topological space to a genus g surface with n boundary components, and locally isomorphic to $\{z \in \mathbb{C} \mid \operatorname{Im} z \geq 0\}$, with its sheaf of holomorphic functions.
(2) A smooth, proper, connected, complex algebraic curve C of genus $2g - 1 + n$, with a real structure, such that $C \setminus C(\mathbb{R})$ has precisely two components, and $C(\mathbb{R})$ consists of n disjoint circles; together with a choice of a component of $C \setminus C(\mathbb{R})$.
(3) Suppose $2g - 2 + n > 0$. Then, a Riemann surface with boundary is equivalently a 2-dimensional connected compact oriented C^∞ manifold Σ with boundary, of genus g with n boundary components, together with a metric of constant curvature -1 such that the boundary is geodesic.

(2) and (3) can be shown to be equivalent (when $2g - 2 + n > 0$) as follows. Given Σ, C is obtained by gluing Σ and $\overline{\Sigma}$ along their boundary. Conversely, given C, Σ is the closure of the chosen component of $C \setminus C(\mathbb{R})$ in C. The hyperbolic metric on Σ is the restriction of the unique complete hyperbolic metric on C compatible with the complex structure.

I will also need nodal Riemann surfaces with boundary. To define these it is easiest to use the algebraic description (2). A nodal Riemann surface with boundary is a proper algebraic curve C, with at most nodal singularities, and a real structure. The real structure on each connected component C_0 of the normalization $\widetilde{C}$ of C must be of the form (2) above; we also require a choice of component of $C_0 \setminus C_0(\mathbb{R})$. All the nodes of C are required to be real, that is in $C(\mathbb{R})$. Let Σ be the closure in C of the chosen components of $C \setminus C(\mathbb{R})$; Σ is a Riemann surface with nodal boundary. Near a node, Σ looks like



The number of boundary components of Σ can be defined as follows. $\partial\Sigma$ will be a union of circles, glued together at points as above. Define a smoothing of $\partial\Sigma$, by replacing each node as above by



The number of boundary components of Σ is defined to be the number of connected components of this smoothing.

Σ has genus g if it has n boundary components and the genus of the nodal algebraic curve $C = \Sigma_{\cup\partial\sigma}\overline{\Sigma}$ is $2g - 1 + n$.

3.2. **The open TCFT operad.** For a finite set I and an integer n, let $\overline{\mathcal{N}}_{g,n,I}$ be the moduli space of Riemann surfaces Σ of genus g with boundary, possibly with nodes on the boundary, with n boundary components, and with I marked points on the boundary. The associated algebraic curve C, obtained from gluing Σ and $\overline{\Sigma}$, must be stable, and of genus $2g - 1 + n$.

Stability is equivalent to the statement that there are only finitely many automorphisms of Σ preserving the marked points. Let $\mathcal{N}_{g,n,I} \subset \overline{\mathcal{N}}_{g,n,I}$ be the locus of non-singular Riemann surfaces (with boundary). These moduli spaces were first constructed by Liu in [13].

Lemma 3.2.1. *$\overline{\mathcal{N}}_{g,n,I}$ is an orbifold with corners of dimension $6g-6+3n+\#I$. The interior of $\overline{\mathcal{N}}_{g,n,I}$ is $\mathcal{N}_{g,n,I}$. The inclusion $\mathcal{N}_{g,n,I} \hookrightarrow \overline{\mathcal{N}}_{g,n,I}$ is a homotopy equivalence.*

The spaces $\overline{\mathcal{N}}_{g,n,I}$ form a modular operad $\overline{\mathcal{N}}$. For an object $[I \twoheadrightarrow J] \in \operatorname{Ob}\mathbf{Graphs}$, define

$$\overline{\mathcal{N}}([I \twoheadrightarrow J]) = \prod_{j\in J}\left(\coprod_{g,n}\overline{\mathcal{N}}_{g,n,I_j}\right)$$

Let $\gamma : [H(\gamma) \twoheadrightarrow V(\gamma)] \to [T(\gamma \twoheadrightarrow C(\gamma)]$ be a map. An element in $\overline{\mathcal{N}}([H(\gamma) \twoheadrightarrow V(\gamma)])$ corresponds to a Riemann surface Σ_v for each vertex $v \in V(\gamma)$, and a marked point on Σ_v for each half-edge at v. Define

$$\overline{\mathcal{N}}(\gamma) : \prod_{v\in V(\gamma)}\left(\coprod_{g,n}\overline{\mathcal{N}}_{g,n,H(v)}\right) \to \prod_{c\in C(\gamma)}\left(\coprod_{g,n}\overline{\mathcal{N}}_{g,n,T(c)}\right)$$

by gluing the disconnected surface $\Sigma = \coprod \Sigma_v$ corresponding to a point in $\prod_{v\in V(\gamma)}\left(\coprod_{g,n}\overline{\mathcal{N}}_{g,n,H(v)}\right)$ to itself, using the edges of γ to identify marked points.

I also need a sub cyclic operad of $\overline{\mathcal{N}}$, which will be identified with Stasheff's topological A_∞ operad. For $[I \twoheadrightarrow J] \in \operatorname{Ob}\mathbf{Forests}$, define

$$A_\infty^{top}([I \twoheadrightarrow J]) = \prod_{j\in J}\overline{\mathcal{N}}_{0,1,I_j}$$

This definition extends in an obvious way to a functor $\mathbf{Forests} \to \mathbf{Top}$, defining a topological cyclic operad A_∞^{top}. The spaces $\overline{\mathcal{N}}_{0,1,I}$ have a natural orientation. As, the open part $\mathcal{N}_{0,1,I} \subset \overline{\mathcal{N}}_{0,1,I}$ can be identified with the quotient of the space of $\#I$ distinct points on the oriented circle S^1 by the action of $\mathrm{PSL}_2(\mathbb{R})$. The natural orientation on $\mathrm{PSL}_2(\mathbb{R})$, coming from it's simply transitive action on the set of unit tangent vectors to the upper half plane, induces an orientation on $\mathcal{N}_{0,1,I}$ and so on $\overline{\mathcal{N}}_{0,1,I}$.

Proposition 3.2.2. *A_∞^{top} is isomorphic as an operad to the topological A_∞ operad of Stasheff.*

Proof. This result is well-known to experts. The compactifications of moduli spaces of marked points on the boundary of a disc used here are the same as those used in Lagrangian Floer homology. This result is therefore the reason for A_∞ relations holding in the Fukaya category. A proof is presented in [1].

I'll briefly sketch a proof. For a cyclically ordered finite set I, let D_I be the compactified moduli space of discs with I marked points on the boundary, such that the natural cyclic order on the boundary coincides with the given one on I. So D_I is a connected component of $\overline{\mathcal{N}}_{0,1,I}$. Fix three consecutive elements $0, 1, \infty \in I$, which we put at $0, 1, \infty$ on the disc. Then, we can identify D_I with a compactification of the space of $I \setminus \{0, 1, \infty\}$ points on the interval. Further, D_I has a cell decomposition, with cells labelled by rooted ribbon-trees which are at least tri-valent. The open cells are given by singular discs of fixed topological type; the root is given by 1, say. Now it is not difficult to identify this cell complex with an associahedron. $\square$

The cyclic operad A_∞^{top} has a cell decomposition, with cells labelled by ribbon forests. The open cells, as before, are given by the surfaces of fixed topological type. The open cells have a natural orientation. This cell decomposition is compatible with the cyclic operad structure. Let A_∞^{alg} be the dg cyclic operad obtained from the cellular chain complexes of A_∞^{top}. A_∞^{alg} is the

standard A_∞ dg cyclic operad, as one can see easily using our earlier description of the latter in terms of forests. The main point is that the boundary of $\overline{\mathcal{N}}_{0,1,n}$, with appropriate orientation, is the sum of copies of $\overline{\mathcal{N}}_{0,1,n_1} \times \overline{\mathcal{N}}_{0,1,n_2}$, where $n_1 + n_2 - 2 = n$, with appropriate signs.

Let C_* be an appropriate chain complex, which has a Künneth map $C_*(X) \otimes C_*(Y) \to C_*(X \times Y)$. Then $C_*(\overline{\mathcal{N}})$ is a dg modular operad.

3.3. **The open TCFT operad and the A_∞ operad.** In this section I show

Theorem 3.3.1. *There is a homotopy equivalence of orbi-space modular operads*

$$\overline{\mathcal{N}} \simeq \mathbf{Mod}(A_\infty^{top})$$

$\mathbf{Mod}(A_\infty^{top})$ is to be considered as an orbi-space. That is, consider A_∞^{top} as a cyclic operad in **Orb**, the 2 category of orbi-spaces, and then apply **Mod**.

An immediate corollary of this theorem is

Corollary 3.3.2. *There is a quasi-isomorphism of dg modular operads over $\mathbb{Q}$,*

$$\mathbf{Mod}(A_\infty^{alg} \otimes \mathbb{Q}) \cong C_*(\overline{\mathcal{N}}) \otimes \mathbb{Q}$$

Markl [15] introduced the notion of *minimal model* of a dg operad. One should be able to generalise this definition to modular operads, and show that $\mathbf{Mod}(A_\infty^{alg} \otimes \mathbb{Q})$ is the minimal model for $C_*(\overline{\mathcal{N}}) \otimes \mathbb{Q})$. Indeed, $\mathbf{Mod}(A_\infty^{alg} \otimes \mathbb{Q})$ is free as a graded modular operad (forgetting the differential), and the image of the differential consists of decomposable elements. Let $(V, \triangle)$ be a complex with an element $\triangle \in V^{\otimes 2}$. Following Markl, one could define a homotopy action of the dg modular operad $C_*(\overline{\mathcal{N}})$ on $(V, \triangle)$ as an action of the minimal model on $(V, \triangle)$. Thus we see, that a action of $C_*(\overline{\mathcal{N}})$ on $(V, \triangle)$ is the same as a homotopy class of homotopy associative structure on $(V, \triangle)$.

The first step in the proof of theorem 3.3.1 is to construct a map of modular operads $\mathbf{Mod}(A_\infty^{top}) \to \overline{\mathcal{N}}$. By the universal property of **Mod**, it is sufficient to give a map of cyclic operads $A_\infty^{top} \to \overline{\mathcal{N}}$; but such a map has already been defined. Let $\Phi : \mathbf{Mod}(A_\infty^{top}) \to \overline{\mathcal{N}}$ be the resulting map of modular operads.

Proposition 3.3.3. Φ *is a homotopy equivalence.*

Let $D_{g,n,I} \subset \overline{\mathcal{N}}_{g,n,I}$ be the locus consisting of curves, with nodes at the boundary, each of whose irreducible components is a disc. One can easily show that $D_{g,n,I} \cong \mathbf{Mod}(A_\infty^{top})$, and that the map $D_{g,n,I} \to \overline{\mathcal{N}}_{g,n,I}$ is just the map Φ described above.

Proposition 3.3.4. *The inclusion $D_{g,n,I} \hookrightarrow \overline{\mathcal{N}}_{g,n,I}$ is a homotopy equivalence.*

This implies proposition 3.3.3. To prove that $D_{g,n,I} \hookrightarrow \overline{\mathcal{N}}_{g,n,I}$ is a homotopy equivalence, it is sufficient to show that

Proposition 3.3.5. *For $(g,n) \neq (0,1)$, the inclusion $\partial\overline{\mathcal{N}}_{g,n,I} \hookrightarrow \overline{\mathcal{N}}_{g,n,I}$ is a homotopy equivalence.*

We will prove this as long as $(g,n) \neq (0,2)$; this case is easy and is left to the reader.

The idea of the proof is very simple. Given a surface $\Sigma \in \mathcal{N}_{g,n,I}$, we use the canonical hyperbolic metric on Σ to flow $\partial\Sigma$ inwards; eventually, Σ becomes singular, and we construct a deformation retract of $\overline{\mathcal{N}}_{g,n,I}$ onto it's boundary.

It is easier to apply this procedure to non-singular surfaces. Therefore, the first step is to move the boundary $\partial\overline{\mathcal{N}}_{g,n,I}$ inwards a little bit.

Let T be a tubular neighbourhood of the boundary $\partial\overline{\mathcal{N}}_{g,n,I}$ which is locally isomorphic to $\partial\overline{\mathcal{N}}_{g,n,I} \times [0,1)$. Let $\mathcal{N}'_{g,n,I} = \overline{\mathcal{N}}_{g,n,I} \setminus T$. $\mathcal{N}'_{g,n,I}$ is a manifold with boundary, and the pair $(\mathcal{N}'_{g,n,I}, \partial\mathcal{N}'_{g,n,I})$ is homotopy equivalent to the pair $(\overline{\mathcal{N}}_{g,n,I}, \partial\overline{\mathcal{N}}_{g,n,I})$. Therefore it is sufficient to show that the inclusion $\partial\mathcal{N}'_{g,n,I} \hookrightarrow \mathcal{N}'_{g,n,I}$ is a homotopy equivalence.

Now we describe a map

$$\Phi : \mathcal{N}'_{g,n,I} \times [0,1] \to \mathcal{N}'_{g,n,I}$$

which is a deformation retraction of the inclusion $\partial\mathcal{N}'_{g,n,I} \hookrightarrow \mathcal{N}'_{g,n,I}$.

The map Φ is constructed as follows. Each surface $\Sigma \in \mathcal{N}'_{g,n,I}$ has a canonical metric of constant curvature -1 with geodesic boundary. Use the exponential map of this metric, and the inward pointing unit normal to $\partial\Sigma \hookrightarrow \Sigma$, to flow the boundary $\partial\Sigma$ inwards. For $t \in \mathbb{R}_{\geq 0}$, let Σ_t be the surface with boundary obtained by flowing in $\partial\Sigma$ a distance t. Eventually, the boundary intersects itself, and we end up with a surface in $\partial\overline{\mathcal{N}}_{g,n,I}$; before this happens, we must have hit $\partial\mathcal{N}'_{g,n,I}$. More precisely,

Lemma 3.3.6. *There is a unique $S \in \mathbb{R}_{\geq 0}$ such that $\Sigma_S \in \partial\mathcal{N}'_{g,n,I}$, and Σ_t is in the interior of $\mathcal{N}'_{g,n,I}$ for all $t < S$.*

The map Φ is now defined by

$$\Phi(\Sigma, x) = \Sigma_{Sx}$$

Proof of lemma. It is sufficient to show that for some T, Σ_T (after forgetting the marked points) is in $\partial\overline{\mathcal{N}}_{g,n,0}$. This will imply that the family of surfaces will have passed through $\partial\mathcal{N}'_{g,n,I}$.

Let T be the first time at which Σ_T is singular. We have to check that $\Sigma_T \in \partial\overline{\mathcal{N}}_{g,n,0}$ (after forgetting the marked points).

By doubling Σ, one can see that for any path $\phi : [0,1] \to \Sigma$, with $\phi(\{0,1\}) \subset \partial\Sigma$, there is a unique geodesic γ, homotopy equivalent to ϕ relative to $\partial\Sigma$, which is normal to $\partial\Sigma$. Further, γ minimises length in this homotopy class.

Let $p_1, p_2 \in \partial\Sigma$ be two distinct points. They collide at time t if there is a point $x \in \Sigma$, and geodesics (p_1, x), (p_2, x) of length t. If the piecewise geodesic $(p_1, x)(x, p_2)$ is not an actual geodesic, then there must be some geodesic γ, in the same homotopy class as $(p_1, x)(x, p_2)$, which is normal to $\partial\Sigma$ and of length shorter than $2t$. This will imply that Σ will have become singular before time t; therefore, at the first time T at which Σ becomes singular, all these piecewise geodesics are smooth.

The time T is the half the minimum length of a geodesic $\gamma : (I, \partial I) \to (\Sigma, \partial\Sigma)$ which is normal to the boundary. In order to show that $\Sigma_T \in \partial\overline{\mathcal{N}}_{g,n,0}$, we have to check 2 things.

(1) There are no three distinct points $p_1, p_2, p_3, \in \partial\Sigma$, which collide at time T and at the same point $x \in \Sigma$. That is, Σ_T has precisely nodal singularities.
(2) Σ_T is stable : there are no irreducible components of Σ_T which are discs with ≤ 2 nodes.

Note that as we forget the marked points of Σ_T, it doesn't matter if they collide with the nodes or each other.

For the first point, suppose $p_1, p_2, p_3 \in \partial\Sigma$ had this property. Then, there is are geodesics (p_i, x) of length T, and we have seen that each piecewise smooth geodesic $(p_i, x)(x, p_j)$ is necessarily smooth. That is, the tangent vectors to each (p_i, x) at x all coincide. This is impossible, as two geodesics which are tangential at any point are the same.

The second point is clear : if we split off a disc with 2 nodes, then we would find two geodesics γ_1, γ_2, normal to the boundary, distinct, and in the same homotopy class. If we split off a disc with one node, then we would find a contractible geodesic γ of positive length, and normal to the boundary.

□

References

[1] Satyan L. Devadoss. Tessellations of moduli spaces and the mosaic operad. In *Homotopy invariant algebraic structures (Baltimore, MD, 1998)*, volume 239 of *Contemp. Math.*, pages 91–114. Amer. Math. Soc., Providence, RI, 1999.

[2] Andreas Floer. Morse theory for Lagrangian intersections. *J. Differential Geom.*, 28(3):513–547, 1988.

[3] Andreas Floer. Witten's complex and infinite-dimensional Morse theory. *J. Differential Geom.*, 30(1):207–221, 1989.

[4] Kenji Fukaya. Floer homology and mirror symmetry. I. In *Winter School on Mirror Symmetry, Vector Bundles and Lagrangian Submanifolds (Cambridge, MA, 1999)*, volume 23 of *AMS/IP Stud. Adv. Math.*, pages 15–43. Amer. Math. Soc., Providence, RI, 2001.

[5] Kenji Fukaya, Yong-Geun Oh, Hiroshi Ohta, and Kaoru Ono. *Lagrangian intersection Floer theory : anomaly and obstruction*. 2000.

[6] E. Getzler and M. M. Kapranov. Cyclic operads and cyclic homology. In *Geometry, topology, & physics*, Conf. Proc. Lecture Notes Geom. Topology, IV, pages 167–201. Internat. Press, Cambridge, MA, 1995.

[7] E. Getzler and M. M. Kapranov. Modular operads. *Compositio Math.*, 110(1):65–126, 1998.

[8] John L. Harer. The virtual cohomological dimension of the mapping class group of an orientable surface. *Invent. Math.*, 84(1):157–176, 1986.

[9] Sheldon Katz and Chiu-Chu Melissa Liu. Enumerative geometry of stable maps with Lagrangian boundary conditions and multiple covers of the disc. *Adv. Theor. Math. Phys.*, 5(1):1–49, 2001.

[10] Ralph M. Kaufmann, Muriel Livernet, and R. C. Penner. Arc operads and arc algebras. *Geom. Topol.*, 7:511–568 (electronic), 2003.

[11] Maxim Kontsevich. Formal (non)commutative symplectic geometry. In *The Gel′fand Mathematical Seminars, 1990–1992*, pages 173–187. Birkhäuser Boston, Boston, MA, 1993.

[12] Maxim Kontsevich. Feynman diagrams and low-dimensional topology. In *First European Congress of Mathematics, Vol. II (Paris, 1992)*, volume 120 of *Progr. Math.*, pages 97–121. Birkhäuser, Basel, 1994.

[13] Chiu-Chu Melissa Liu. Moduli of J-holomorphic curves with Lagrangian boundary conditions and open Gromov-Witten invariants for an S^1-equivariant pair. 2002.

[14] Riccardo Longoni and Thomas Tradler. Homotopy inner products for cyclic operads. 2003.

[15] Martin Markl. Models for operads. *Comm. Algebra*, 24(4):1471–1500, 1996.

[16] Martin Markl, Steve Shnider, and Jim Stasheff. *Operads in algebra, topology and physics*, volume 96 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2002.

[17] Gregory Moore. Some comments on branes, G-flux, and K-theory. In *Strings 2000. Proceedings of the International Superstrings Conference (Ann Arbor, MI)*, volume 16, pages 936–944, 2001.

[18] R. C. Penner. The decorated Teichmüller space of punctured surfaces. *Comm. Math. Phys.*, 113(2):299–339, 1987.

[19] R.C. Penner. Cell decomposition and compactification of Riemann's moduli space in decorated Teichmüller theory. 2003.

[20] Graeme Segal. Topological structures in string theory. *Phil. Trans. R. Soc. Lond. A*, (359):1389–1398, 2001.

[21] James Dillon Stasheff. Homotopy associativity of H-spaces. I, II. *Trans. Amer. Math. Soc. 108 (1963), 275-292; ibid.*, 108:293–312, 1963.

[22] Thomas Tradler. Infinity-inner-products on A-infinity-algebras. 2001.

[23] Edward Witten. Noncommutative geometry and string field theory. *Nuclear Phys. B*, 268(2):253–294, 1986.

[24] Edward Witten. Chern-Simons gauge theory as a string theory. In *The Floer memorial volume*, volume 133 of *Progr. Math.*, pages 637–678. Birkhäuser, Basel, 1995.

DEPARTMENT OF MATHEMATICS, IMPERIAL COLLEGE LONDON, LONDON SW7 2AZ, UNITED KINGDOM
E-mail address: k.costello@imperial.ac.uk